Problem Set - Theory of Economics of Conflicts
LMU, Summer Semester 2018

Please, take your solutions of the Problem Set at the lecture on the 12th of June, so that you will be able to check if your solutions are correct.

Consider a *generalized Tullock contest* with complete information; that is, the probability of victory of player $i \in N$ where $N = \{1, ..., n\}$ is the set of players equals

$$p_i(x_1, ..., x_n) = \frac{x_i^r}{\sum_{j \in N} x_j^r}$$

where x_i and x_j are efforts of player i and j, and assume $r > 0$. Each player i values the prize $V_i > 0$ and the marginal cost of effort equals 1 for every player.

1. Consider $n = 2$, $r \leq 1$ and possibly different valuations V_1 and V_2. Show that the second-order conditions (SOC) hold, and thus find the equilibrium efforts (x_1^*, x_2^*). Analyse which player is more likely to win the prize in equilibrium. Derive the equilibrium payoff for each player.

2. Consider $n = 2$, $r \leq 1$, $V_1 \geq V_2$, so that the equilibrium efforts found in 1. continue to hold. Show that the sum of efforts is greater than $r\frac{V_2}{2}$. (Hint: once you have the expression for the sum of efforts, notice that both $\frac{1+x}{1+x^r}$ and $\frac{x^r}{1+x^r}$ are increasing in x when $x \geq 1$ and $r \in (0, 1]$)

3. Consider $n \geq 2$, $r \leq 1$ and $V_1 = V_2 = ... = V_n = V$. Show that the equilibrium individual effort decreases in n and increases in r and V, that the sum of efforts increases in n, r and V, and that full rent dissipation is achieved only when $r = 1$ and $n \to \infty$.

4. Consider $n = 2$, and possibly different valuations V_1 and V_2. Show that $r \leq 1 + \left(\frac{V_1}{V_2}\right)^r$ suffices for SOC to hold, so that the equilibrium expressions found in 1. continue to hold. (Hint: claim that in equilibrium $x_1^* > 0$ and $x_2^* > 0$, so that the first-order condition (FOC) pinpoint the equilibrium if the SOC hold *under the FOC*. In other words, the unique maximum characterized by the FOC is global)

Consider a *best-of-three Tullock contest* under complete information; that is, two players play at most three matches, and the first player who wins two matches is the contest winner and obtains a prize equal to $V > 0$. There is no intermediate prize; that is, the only prize at stake is $V > 0$ which is given to the first player who wins two matches. In each match, players exert efforts for that match (x_1 for player 1 and x_2 for player 2), and player $i \in \{1, 2\}$ wins that match with probability

$$p_i(x_1, x_2) = \frac{x_1}{x_1 + x_2}$$

and pays a cost of effort equal to x_i. In such a game, compute the equilibrium efforts of each match.

(Hint #1: being a sequential game, apply backward induction. Thus, start from the third tie-breaking match (if any), and compute equilibrium efforts and payoffs. These equilibrium payoffs determine what players are fighting for in the second match. In turn, second-match equilibrium payoffs determine what players are fighting for in the first match)